IWMS'2016
Madeira Portugal

AUT

A 50 - year journey with colleagues, generalized matrix inverses and applied probability

Jeffrey J Hunter

Mathematical Sciences, Auckland University of Technology, New Zealand

1965: In the beginning.....

In Feb 1965 introduced to "conditional inverses" in a graduate course Statistics 150 "Analysis of Variance with application to Experimental Design" within the Dept of Statistics at the UNC Chapel Hill taught by Indra Chakravarti, using notes prepared by Raj Chandra Bose:

Let $A = A(m \times n)$ *be any matrix.*

Then $A^* = A^*(n \times m)$ *will be defined to be a conditional inverse of* A, *if* $AA^*A = A$.

If $A^* = A^*(n \times m)$ *is a conditional inverse of* $A = A(m \times n)$ *and if the equations* $Ax = y$ *are consistent then* x_1 *is a solution if* $x_1 = A^*y$.

1968: On the renewal density matrix of a semi-Markov process

This was the topic of my Ph.D. thesis from the University of North Carolina at Chapel Hill which examined the necessary and sufficient conditions for the convergence of the renewal density matrix.

My supervisor was Prof Walter L Smith who taught me a course on "Stochastic Processes" that whetted my appetite to work in applied probability, in particular Markov renewal processes, Markov chains, queueing theory, & renewal theory.

"The mathematics that lasts is the mathematics that is beautiful" (W.L.Smih)

1969: Moments of Markov Renewal Processes

This paper contained for the first time the identification of $Z = [I - P + \Pi]^{-1}$, Kemeny & Snell's fundamental matrix of a finite irreducible MC as a (one condition) generalized matrix inverse of $I - P$.

The paper contains what Marcel Neuts referred to as the "classical result on the mean recurrence time in an irreducible positive recurrent Markov renewal process".

Advances in Applied Probability, **1** (2), 188 - 210, (1969).

1969: Moments of Markov Renewal Processes

Key Results:

$$m_{ij}^{(r)} = \sum_{k\neq j} p_{ik} m_{kj}^{(r)} + \sum_{s=1}^{r-1} \binom{r}{s} \left\{ \sum_{k\neq j} \mu_{ik}^{(r-s)} m_{kj}^{(s)} \right\} + \mu_i^{(r)}.$$

$Z = [I - P + \Pi]^{-1}$ is a 1-condition generalized inverse of $I - P$.

$$M = \left[m_{ij}^{(1)} \right] = \left[\frac{1}{\lambda_1} \left\{ ZP^{(1)}\Pi - E(ZP^{(1)}\Pi)_d \right\} + I - Z + EZ_d \right] D.$$

Expression for $M^{(2)} = \left[m_{ij}^{(2)} \right]$.

$$\boldsymbol{M}(t) = \left[E\left\{ N_j(t) \middle| Z_0 = i \right\} \right] = \frac{t}{\lambda_1}\Pi + \frac{\lambda_2}{2\lambda_1^2}\Pi + \left[\frac{1}{\lambda_1}\Pi P^{(1)} - I \right] Z \left[\frac{1}{\lambda_1} P^{(1)}\Pi - I \right] - I + o(1)E.$$

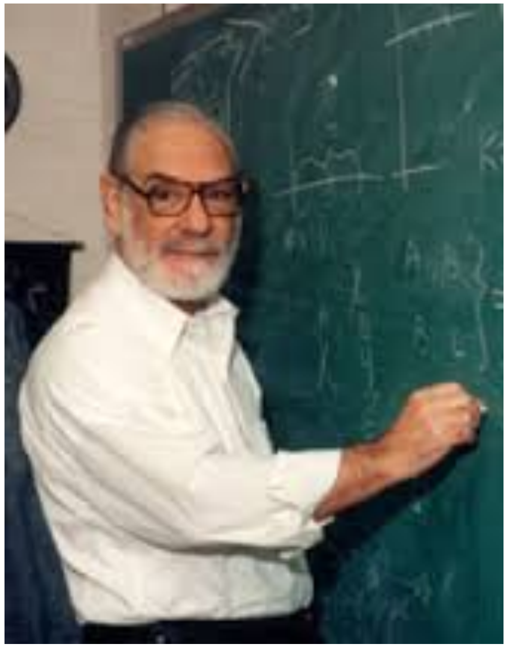
Expression for $W(t) = \left[E\left\{ N_j(t)(N_j(t)+1\} \middle| Z_0 = i \right\} \right] = A_1 t^2 + A_2 t + A_1 + o(1)E.$

Advances in Applied Probability, **1** (2), 188 - 210, (1969).

1973: Sabbatical Leave at UNC-CH

Conference on Analytic and Algebraic Methods in Queueing Theory, Univ Western Michigan, Kalamazoo, Michigan (May).

Met Ralph Disney who invited me to Univ Michigan (June) to speak on my current research on "Renewal theory in two dimensions". (Led eventually to 3 papers in the Advances in Applied Probability). Led to visits with Disney in 1980, 1987, and 1992 in NZ.



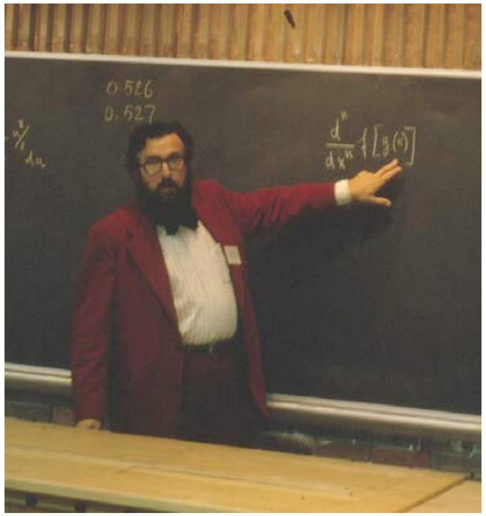
1973: Sabbatical Leave at UNC-CH

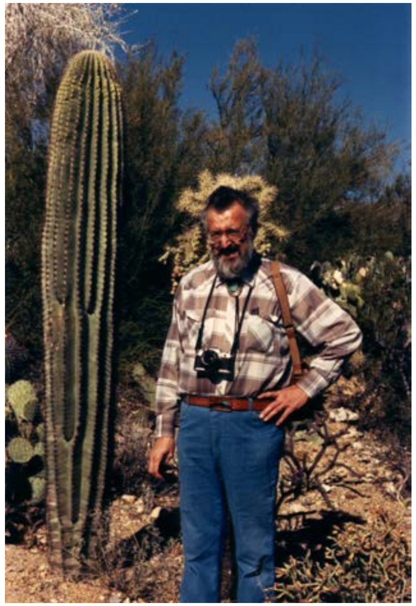
Conference on Analytic and Algebraic Methods in Queueing Theory, Univ Western Michigan, Kalamazoo, Michigan (May).

Met Marcel Neuts who invited me to Purdue Univ (June).

Later to visit him at Univ. of Arizona in 1988

and Marcel to NZ in 1990.





1973: Sabbatical Leave at UNC-CH

Eastern Regional Meeting, Institute of Mathematical Statistics, Ithaca College, Ithaca, (May 30-June 1).

Met George Styan for the first time.

(George was talking on MCs and using generalized matrix inverses.)

Visit to McGill 1988 following George's visit to Auckland 1985.

1975: Don McNickle PhD

Donald C. McNickle completed his Ph.D. in Mathematics, Univ. of Auckland, under my supervision.

Thesis title:
"Processes in the decomposition of networks of queues”.

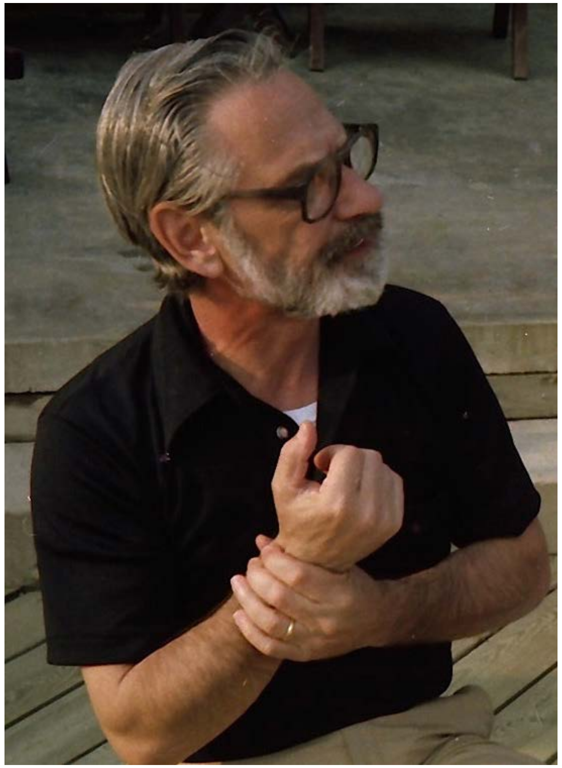
Don took up a post-doctoral position with Ralph Disney.

He recently retired from the Dept. Management, Univ. of Canterbury.

1980: College of Engineering Visiting Prof within Dept IEOR at VPI&SU

Hosted by Ralph Disney. Taught graduate course on "Queueing Theory" & worked with his graduate students.

Completed a major research paper on“ Generalized inverses and their applications to applied probability problems”.



1982: Generalized inverses and their applications to applied probability problems

Examines the applicability of generalized inverses to a wide variety of problems where a MC is present.

Let P be the transition natrix of a finite irreducible MC with stationary probability vector $\boldsymbol{\pi}^T$.

Let $\boldsymbol{u}$ and $\boldsymbol{t}$ be such that $\boldsymbol{u}^T\boldsymbol{e} \neq 0$ and $\boldsymbol{\pi}^T\boldsymbol{t} \neq 0$. Then $I - P + \boldsymbol{tu}^T$ is nonsingular and $[I - P + \boldsymbol{tu}^T]^{-1}$ is a g-inverse of $I - P$.

Any g-inverse of $I - P$ has the form G = $[I - P + \boldsymbol{tu}^T]^{-1} + \boldsymbol{ef}^T + \boldsymbol{g\pi}^T$.

Obtained general procedures for finding stationary distributions, moments of the first passage time distributions, and asymptotic forms for the moments of the occupation-time random variables.

Linear Algebra and its Applications, **45**, 157 - 198, (1982)

1982: Generalized inverses and their applications to applied probability problems

In particular, let G be any g-inverse of $I-P$.

$$\boxed{\text{If } A = I-(I-P)G \text{ then } \boldsymbol{\pi}^T = \frac{\boldsymbol{v}^T \mathrm{A}}{\boldsymbol{v}^T \mathrm{A}\boldsymbol{e}} \text{ for any } \boldsymbol{v}^T \text{ such that } \boldsymbol{v}^T \mathrm{A}\boldsymbol{e} \neq \boldsymbol{0}.}$$

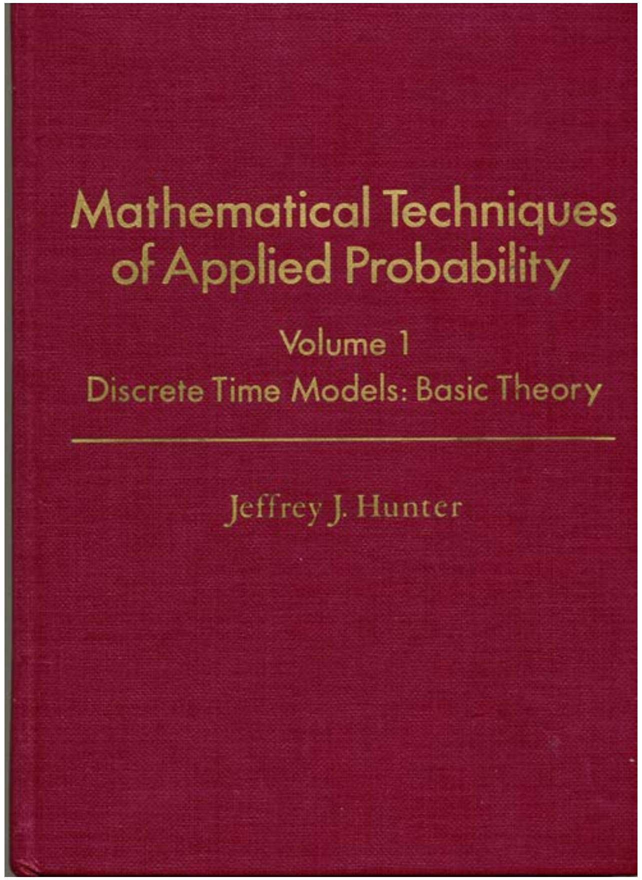
$$\boxed{M = [G\Pi - E(G\Pi)_d + I - G + EG_d]D \text{ where } D = (\Pi_d)^{-1}.}$$

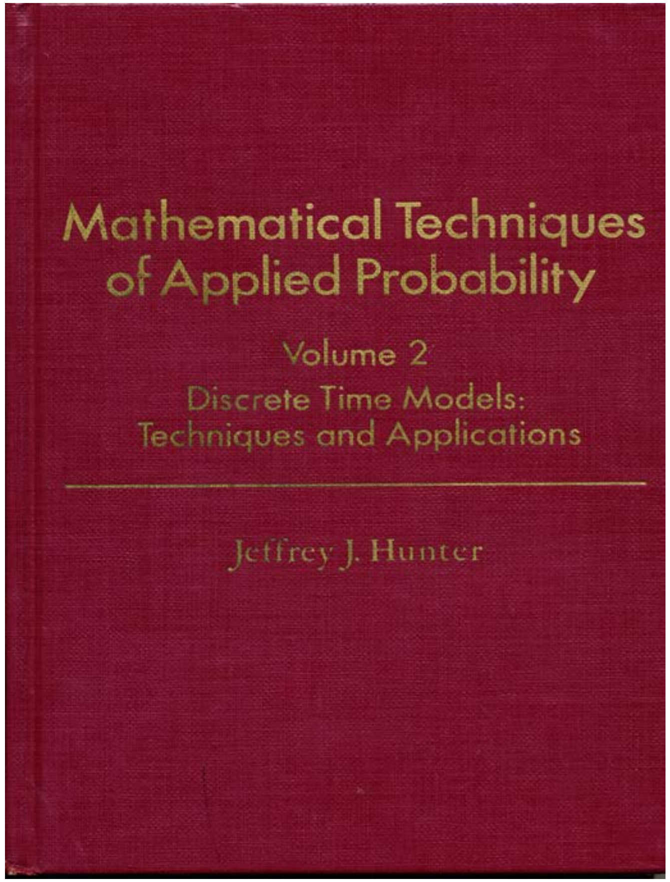
$$\boxed{\left[EN_{ij}^{(n)}\right] = (n+1)\Pi - I + (I-\Pi)G(I-\Pi) + o(1)E.}$$

The above problems are also examined for MRPs and MCs in continuous time. In the latter case we can use g- inverses of $I-P$, or g-inverses of the infinitesimal generator of the process .

Linear Algebra and its Applications, **45**, 157 - 198, (1982)

1983: “Mathematical Techniques of Applied Probability”





Volumes 1 and 2 published by Academic Press

1986: Stationary distributions of perturbed Markov chains

Techniques for updating the stationary distribution of a finite irreducible MC following a rank one perturbation of its transition matrix were discussed.

Let $P^{(2)} = P^{(1)} + \boldsymbol{a}\boldsymbol{b}^T$ with $\boldsymbol{b}^T\boldsymbol{e} = 0$.

Choose $\boldsymbol{u}$ and $\boldsymbol{t}$ (with $\boldsymbol{u}^T\boldsymbol{e} \neq 0$, $\boldsymbol{\pi}^{(1)T}\boldsymbol{t} \neq 0$).

Let $\boldsymbol{\alpha}^T = \boldsymbol{u}^T[I - P^{(1)} + \boldsymbol{t}\boldsymbol{u}^T]^{-1}$ and $\boldsymbol{\beta}^T = \boldsymbol{b}^T[I - P^{(1)} + \boldsymbol{t}\boldsymbol{u}^T]^{-1}$.

Then $\boldsymbol{\pi}^{(1)T} = \dfrac{\boldsymbol{\alpha}^T}{\boldsymbol{\alpha}^T\boldsymbol{e}}$ and $\boldsymbol{\pi}^{(2)T} = \dfrac{(\boldsymbol{\alpha}^T\boldsymbol{a})\boldsymbol{\beta}^T + (1 - \boldsymbol{\beta}^T\boldsymbol{a})\boldsymbol{\alpha}^T}{(\boldsymbol{\alpha}^T\boldsymbol{a})(\boldsymbol{\beta}^T\boldsymbol{e}) + (1 - \boldsymbol{\beta}^T\boldsymbol{a})(\boldsymbol{\alpha}^T\boldsymbol{e})}$.

A variety of situations where such perturbations may arise are presented together with suitable procedures for the derivation of the related stationary distributions.

Linear Algebra and its Applications, 82, 201 - 214, (1986)

1987: 16th Conference on Stochastic Processes and their Applications, Stanford

Met with Bill Henderson, Mike Rumsewicz, Peter Taylor

1987: Sabbatical Leave at VPI & SU

Taught a course on TV on "Analysis of Queueing Systems" that was beamed over the Eastern U.S. with students at 6 different locations.

Pursued research on feedback queueing systems. (Led to 4 papers on Filtering of Markov Renewal queues – Advances in Applied Probability).

1988: Characterizations of generalized inverses associated with Markovian kernels

Conditions: (1) $AXA = A$, (2) $XAX = X$, (3) $(AX)^T = AX$,

(4) $(XA)^T = (XA)$, (5) For square matrices $AX = XA$.

Characterisations of A{1,2}, A{1,3}, A{1,4}, A{1,5}, A{1, j, k}.

Partitioned forms for the g-inverses are also presented based on a full-rank factorization of I - P. Special well-known cases.

Group inverse: $\text{A}\{1,2,5\} = [I - P + \boldsymbol{e}\boldsymbol{\pi}^T]^{-1} - \boldsymbol{e}\boldsymbol{\pi}^T$

$= (I - \boldsymbol{e}\boldsymbol{\pi}^T)G(I - \boldsymbol{e}\boldsymbol{\pi}^T)$ for any g.i. G of I - P.

Moore-Penrose inverse:

$\text{A}\{1,2,3,4\} = [I - P + \alpha\boldsymbol{\pi}\boldsymbol{e}^T]^{-1} - \alpha\boldsymbol{e}\boldsymbol{\pi}^T$, $\alpha = (m\pi^T\pi)^{-1/2}$.

Linear Algebra and its Applications, 102, 121 - 142, (1988)

1988: Characterizations of generalized inverses associated with Markovian kernels

A systematic investigation of the various multi-condition generalized inverses of $I-P$, where P is the transition matrix of a finite, irreducible, discrete-time MC chain, is presented.

If $\boldsymbol{u}_i$ and $\boldsymbol{t}_i$ such that $\boldsymbol{u}_i^T\boldsymbol{e} \neq 0$ and $\boldsymbol{\pi}^T\boldsymbol{t}_i \neq 0$, then

$$[I-P+\boldsymbol{t}_2\boldsymbol{u}_2^T]^{-1} = (I-U_2)[I-P+\boldsymbol{t}_1\boldsymbol{u}_1^T]^{-1}(I-T_2)+\beta_{22}\boldsymbol{e}\boldsymbol{\pi}^T,$$

where $U_2 = \dfrac{\boldsymbol{e}\boldsymbol{u}_2^T}{\boldsymbol{u}_2^T\boldsymbol{e}}$, $T_2 = \dfrac{\boldsymbol{t}_2\boldsymbol{\pi}^T}{\boldsymbol{\pi}^T\boldsymbol{t}_2}$, and $\beta_{22} = \dfrac{1}{(\boldsymbol{\pi}^T\boldsymbol{t}_2)(\boldsymbol{u}_2^T\boldsymbol{e})}$.

$$A_\delta \equiv [I-P+\delta \boldsymbol{t}\boldsymbol{u}^T]^{-1} - \frac{\boldsymbol{e}\boldsymbol{\pi}^T}{\delta(\boldsymbol{\pi}^T\boldsymbol{t})(\boldsymbol{u}^T\boldsymbol{e})}$$ does not depend on δ.

Based on results of Greville and Ben-Israel.

Linear Algebra and its Applications, 102, 121 - 142, (1988)

1988: Visiting Research Scholar at Centre for Stochastic Processes, UNC-CH

Hosted by Ross Leadbetter (Jan - March).

Pursued research on computing the stationary distribution of a Markov chain that lead to a publication in 1991.

1988: Visit George Styan at McGill

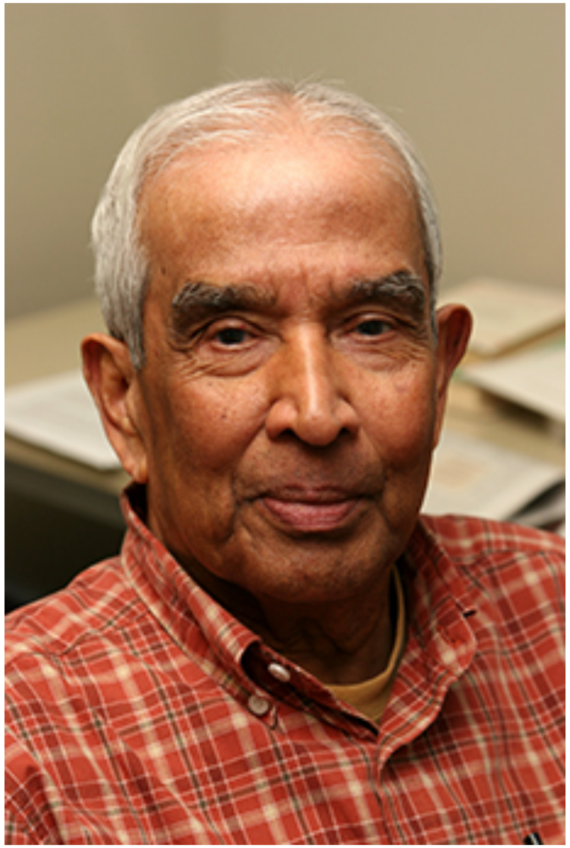
Pursued research (April) on "parametrising" generalized inverses which lead to a publication in 1990.

George tried to persuade me to stop research on generalised inverses!

1988: Visiting Scientist, Math Sciences Institute at Cornell University

Hosted by Uma Prabhu during August. Uma had appointed me as an Associate Editor of QUESTA.

I pursued research on "Sojourn time problems in feedback queues" which was presented, at that time, at a "Workshop on Mathematical Theory of Queueing Systems", held at Cornell.



QUESTA: Queueing Systems, Theory and Applications, **5**, 55 - 76, (1989)

1990: Parametric forms for generalized inverses of Markovian kernels and their applications

Parametric forms for multi-condition generalized inverses of $I-P$, where P is the transition matrix of a finite irreducible discrete time MC, are derived.

Given any g-inverse, G, of $I - P$ there exist unique parameters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and γ such that

$G = [I - P + \boldsymbol{\alpha}\boldsymbol{\beta}^T]^{-1} + \gamma \boldsymbol{e}\boldsymbol{\pi}^T.$

Let $A = I-(I-P)G$, $B = I - G(I-P)$.

Then $\boldsymbol{\alpha} = Ae$, $\boldsymbol{\beta}^T = \boldsymbol{\pi}^T B$, $\gamma + 1 = \boldsymbol{\beta}^T G\boldsymbol{\alpha}.$

Note $\gamma + 1 = \boldsymbol{\pi}^T G\boldsymbol{\alpha} = \boldsymbol{\beta}^T G\boldsymbol{e}, \boldsymbol{\pi}^T\boldsymbol{\alpha} = 1, \boldsymbol{\beta}^T\boldsymbol{e} = 1.$

Linear Algebra and its Applications, 127, 71 - 84, (1990)

1990: Parametric forms for generalized inverses of Markovian kernels and their applications

$$G \in A\{1,2\} \Leftrightarrow \gamma = -1,$$

$$G \in A\{1,3\} \Leftrightarrow \boldsymbol{\alpha} = \pi / \boldsymbol{\pi}^{T} \boldsymbol{\pi},$$

$$G \in A\{1,4\} \Leftrightarrow \boldsymbol{\beta} = \boldsymbol{e}^{\mathrm{T}} / \boldsymbol{e}^{\mathrm{T}} \boldsymbol{e},$$

$$G \in A\{1, 5\} \Leftrightarrow \boldsymbol{\alpha} = \boldsymbol{e},\ \boldsymbol{\beta} = \boldsymbol{\pi}.$$

These are utilized in techniques for obtaining moments of first passage time distributions.

Linear Algebra and its Applications, 127, 71 - 84, (1990)

1991: The computation of stationary distributions of Markov chains through perturbations

Let $G_0 = I$, $\boldsymbol{u}_0^T = \mathbf{e}^T \big/ m$.

For $i = 1, 2, \ldots, m$, let $\boldsymbol{p}_i^T = \mathbf{e}_i^T P$,

$$\boldsymbol{u}_i^T = \boldsymbol{u}_{i-1}^T + \boldsymbol{p}_i^T - \mathbf{e}^T \big/ m,$$

$$G_i = G_{i-1} + G_{i-1}(\mathbf{e}_{i-1} - \mathbf{e}_i)(\boldsymbol{u}_{i-1}^T G_{i-1} \big/ \boldsymbol{u}_{i-1}^T G_{i-1} \mathbf{e}_i).$$

At $i = m$, let $G = G_m$ and

$$\boldsymbol{\pi}^T = \boldsymbol{\pi}_m^T = \frac{\boldsymbol{u}_m^T G_m}{\boldsymbol{u}_m^T G_m \mathbf{e}}.$$

Journal of Applied Mathematics and Stochastic Analysis, 4 (1), 29 - 46, (1991)

1992: 2nd IWMS, Auckland

1992: Stationary distributions and mean first passage times in MCs using generalized inverses

The joint derivation of stationary distributions and mean first passage times using various g-inverses:

1. Compute $G = [g_{ij}]$, a g-inverse of $I - P$.
2. Compute, sequentially, rows 1, 2, …r ($\leq$ m) of

   $A = I - (I - P)G \equiv [a_{ij}]$ until $\sum_{k=1}^{m} a_{rk}$, (1$\leq$ r $\leq$ m)

   is the first non-zero sum.
3. Compute $\pi_j = a_{rj} \Big/ \sum_{k=1}^{m} a_{rk}, \quad j = 1, \ldots, m.$
4. Compute $m_{jj} = \sum_{k=1}^{m} a_{rk} \Big/ a_{rj}, j = 1, \ldots, m,$ and for, $i \neq j$.

   $$m_{ij} = \left\{(g_{jj} - g_{ij})\sum_{k=1}^{m} a_{rk} \Big/ a_{rj}\right\} + \left\{\sum_{k=1}^{m} (g_{ik} - g_{jk})\right\}.$$

Asia - Pacific Journal Operational Research, 9 (2), 145 - 153, (1992)

1999: International Conf on Stochastic Processes & their Applications, Cochin Univ of Science and Technology, Cochin, India

Invited by Prof Krishnamoorthy as a Plenary Speaker

Talk on "A Survey of Generalized Inverses and their use in Stochastic Modelling". Published as a book chapter.

Advances in Probability and Stochastic Processes, A Volume in Honor of Professors R.P. Pakshirajan, G. Sankaranarayanan & S.K. Srinivasan, Notable Publications Inc., New Jersey, USA

2001: Sabbatical Leave at UNC-CH

Visit to Carl Meyer at NC State

We discussed the importance of mean first passage times in relation to the effects on the stationary probabilities when the MC transition matrix is subjected to perturbations.

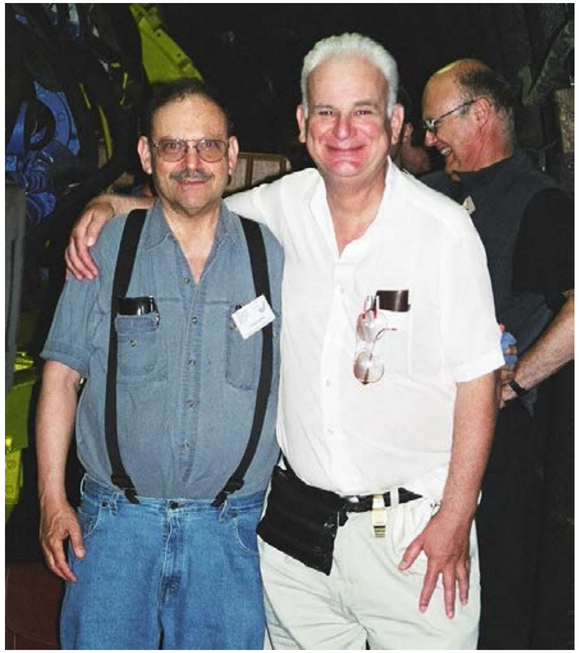
2002: Visit to Univ of Nottingham

Met with Robin Milne (Univ of Western Australia) and Frank Ball (Univ of Nottingham).

Both have quoted my results on the moments of MRPs and used the results in modelling ion channels.

2003: 12th IWMS, Dortmund, Germany

Some colleagues – Jochen, Adi, George



2004: 13th IWMS, Bedlewo, Poland

2005: Stationary distributions and mean first passage times of perturbed Markov chains

The mean first passage times play an important role in determining the differences between the stationary probabilities in the perturbed and unperturbed situations.

Let G be any g-inverse of $I-P$. Let $H = G(I-P)$ then,

for any general perturbation $\mathbf{E}$, $\tilde{\boldsymbol{\pi}}^T - \boldsymbol{\pi}^T = \tilde{\boldsymbol{\pi}}^T \mathbf{E} H$.

Let $N = [n_{ij}] = [(1-\delta_{ij})m_{ij} / m_{jj}] = [(1-\delta_{ij})m_{ij}\pi_j]$ then,

for any general perturbation $\mathbf{E}$, $\tilde{\boldsymbol{\pi}}^T - \boldsymbol{\pi}^T = -\tilde{\boldsymbol{\pi}}^T \mathbf{E} N$.

If $\boldsymbol{E} = [\varepsilon_{ij}]$ and $\alpha_l = \sum_{k=1}^{m} \tilde{\pi}_k \varepsilon_{kl}$ then $\pi_j - \tilde{\pi}_j = \sum_{l \neq j} \alpha_l \, n_{lj}$.

Some bounds and special cases of row perturbations explored.

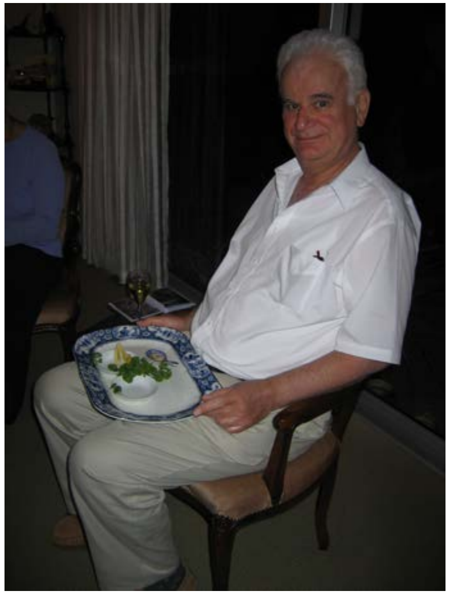
Also updating mean first passage times under perturbations.

Linear Algebra and its Applications, 410, 217 - 243, (2005)

2005: Chair LOC 14th IWMS, Auckland

2005: 14th IWMS, Auckland

2005: Meals at Chateau Hunter



2005: University of Canberra Statistics Workshop, Canberra, Australia

2006: Perturbed Markov chains

Exploration of relative and absolute differences between the stationary probabilities in the perturbed and unperturbed situations. Bounds for these differences are given and are illustrated by means of an 8 - state MC example.

If the transition probability p_{ra} in an irreducible finite MC is decreased by an amount ε while p_{rb} is increased by an amount ε then, with the irreducibility preserved,

$$\left|\frac{\pi_j - \tilde{\pi}_j}{\pi_j}\right| \leq \varepsilon \tilde{\pi}_r \max\{m_{ab}, m_{ba}\} = \max\left\{\left|\frac{\pi_a - \tilde{\pi}_a}{\pi_a}\right|, \left|\frac{\pi_b - \tilde{\pi}_b}{\pi_b}\right|\right\}.$$

The example shows how difficult it is to establish universal results predicting how MCs behave under perturbations.

In Peter Brown, Shuangzhe Liu and Dharmendra Sharma (Eds.), *Contributions to Probability and Statistics - Applications and Challenges: Pro International Statistics Workshop , Univ of Canberra, 2005* World Scientific

2006: 15th IWMS, Uppsala

2006: Mixing times with applications to perturbed Markov chains

A measure of the “mixing time” or “time to stationarity” in a finite irreducible discrete time MC is considered. The statistic $\eta_i = \sum_{j=1}^{m} m_{ij}\pi_j$, is shown to be independent of the initial state i (so that $\eta_i = \eta$ for all i), is minimal in the case of a periodic MC, yet can be arbitrarily large in a variety of situations.

If $G = [g_{ij}]$ is any g-inverse of $I - P$, $\eta = 1 + \sum_{j=1}^{m}(g_{jj} - g_{j.}\pi_j)$.

For any irreducible m-state MC, $\eta \geq \dfrac{m+1}{2}$.

For all irreducible m-state MCs undergoing a general perturbation $\boldsymbol{E} = [\varepsilon_{ij}]$

$\left\| \boldsymbol{\pi} - \bar{\boldsymbol{\pi}} \right\|_1 \leq (\eta - 1) \left\| \boldsymbol{E} \right\|_\infty$ where $\left\| \boldsymbol{E} \right\|_\infty = \max_{1\leq k\leq m} \sum_{l=1}^{m} |\varepsilon_{kl}|$.

Linear Algebra and its Applications, 417, 108-123, (2006)

2006: Workshop on “Matrix Theory and Applications in Physical, Biological and Social Sciences”, Penn State Univ, PA, USA

2007: 16th IWMS, Windsor, Canada

2007: Canadian Statistical Society, St Johns, Newfoundland

2007: Simple procedures for finding mean first passage times in Markov chains

Elegant new results for finding the mean first passage times.

The procedures of this paper involve only the derivation of the inverse of a matrix of simple structure, based upon known characteristics of the Markov chain together with simple elementary vectors. No prior computations are required.

Various possible families of matrices are explored leading to different related procedures.

If $G_{eb} = [I - P + \boldsymbol{e}\boldsymbol{e}_b^T]^{-1} = [g_{ij}]$. then

$$\pi_j = g_{bj},\ j = 1,2,\ldots,m, \text{ and } m_{ij} = \begin{cases} 1/g_{bj}, & i = j, \\ (g_{jj} - g_{ij})\big/ g_{bj}, & i \neq j. \end{cases}$$

Asia - Pacific Journal Operational Research, 24 (6), 813-829, (2007)

2008: 17th IWMS Tomar, Portugal

2008: Variance of first passage times and applications to mixing times in Markov chains

A study of the computation of second moments of the mixing times, and the variance of the first passage times, in a discrete time MC is carried out leading to some new results.

If G is any g-inverse of $I-P$, then

$$M^{(2)} = 2[GM - E(GM)_d] + 2[I - G + EG_d]D(\Pi M)_d - M.$$

We explore the variance of the mixing times, starting in state i. They are shown to depend on i and an exploration of recommended starting states is considered.

$$\boldsymbol{\nu} = \eta^{(2)} - \eta^2\boldsymbol{e} = 2[I - G + EG_d]\boldsymbol{\alpha} + [2\eta G - 2tr(GL) - \eta - \eta^2]\boldsymbol{e},$$

where $\boldsymbol{\alpha}^T = \boldsymbol{\pi}^T M = \boldsymbol{e}^T(\Pi M)_d$, $L = M\Pi_d = [m_{ij}\pi_j]$ (mixing matrix).

Linear Algebra and its Applications, 429, 1135-1162, (2008)

2009: 19th IWMS, Smolenice, Slovakia

2009: Coupling and mixing times in a Markov chain

The derivation of the expected time to coupling in aMC and its relation to the expected time to mixing are explored.

Suppose $\{X_n\}$ *and* $\{Y_n\}$ both have state space $S = \{1,2, \ldots m\}$,

$Z_n = (X_n, Y_n)$, $(n \geq 0)$, is a 2-dim MC with state space $S \times S$.

$C = \{(i,i), 1 \leq i \leq m\}$ are the coupling states.

$T_{ij,C}$ be the first passage time from (i, j), $(i \neq j)$ to C.

$\kappa_{ij}^{(C)} = E[T_{ij,C}]$ (coupling occurs with probability one)

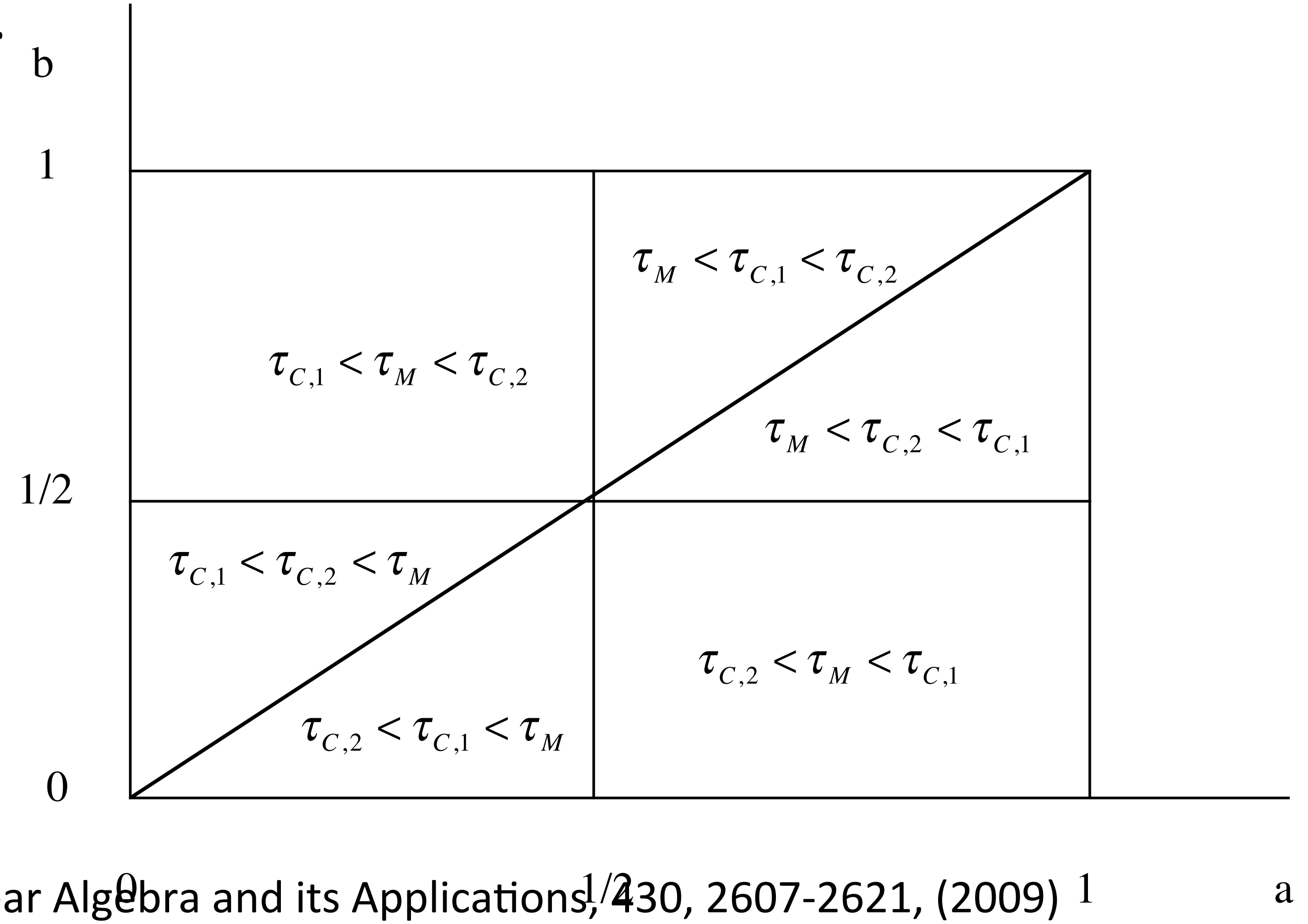
$\tau_{C,i} = \sum_{j=1}^{m} \pi_j E[T_{ij,C}] = \sum_{j \neq i} \pi_j \kappa_{ij}^{(C)} =$ expecting time to coupling from i

Computation of $\tau_{C,i}$ discussed in detail.

Linear Algebra and its Applications, 430, 2607-2621, (2009)

2009: Coupling and mixing times in a Markov chain

The two-state cases and three-state cases are examined in detail.



Linear Algebra and its Applications, 430, 2607-2621, (2009)

2009: Bounds on Expected Coupling Times in a Markov Chain

In the paper "Coupling and Mixing Times in MarkovChains" it was shown that it is very difficult to find explicit expressions for the expected time to couplingin a general Markovchain. Simple upper and lower bounds are given for the expected time to coupling in a discrete time finite Markov chain. Detailed comparisons are provided for two and three state chains.

"Statistical Inference, Econometric Analysis and Matrix Algebra. Festschrift in Honour of Götz Trenkler", (pp*271-294),*
Bernhard Schipp and Walter Krämer (Editors), Physica-Verlag Heidelberg.
ISBN 978-3-7908-2120-8, e-ISBN: 978-3-7908-2121-5

2010: Chair IOC 20th IWMS, Shanghai

2010: Some stochastic properties of semi-magic and magic Markov chains

This paper explores the main stochastic properties of "magic" MCs (formed from scaled magic squares) as well as "semi-magic" MCs (with doubly-stochastic transition matrices). Stationary distributions, generalized inverses of Markovian kernels, mean first passage times, variances of the first passage times and expected times to mixing are considered. Some general results are developed, some observations from the MCs generated by MATLAB are discussed, some conjectures are presented and some special cases, involving three and four states, are explored in detail.

Linear Algebra and its Applications, 433, 893-907, (2010)

2010: Some stochastic properties of semi-magic and magic Markov chains

For any order n doubly stochastic matrix $\lim_{k\to\infty} P^k = \frac{\boldsymbol{e}\boldsymbol{e}^T}{\boldsymbol{e}^T\boldsymbol{e}} = \frac{1}{n}E.$

$\boldsymbol{\pi}^T = \frac{\boldsymbol{e}^T}{n} \Leftrightarrow P$ is doubly stochastic.

$A^{\#} = A^{\dagger} \Leftrightarrow P$ is doubly stochastic.

For 3-state doubly stochastic P, M is semi-magic

$$P = \begin{bmatrix} 1-b-c & b & c \\ c & 1-b-c & b \\ b & c & 1-b-c \end{bmatrix} \Rightarrow M = \begin{bmatrix} 3 & \frac{b+2c}{\delta} & \frac{2b+c}{\delta} \\ \frac{2b+c}{\delta} & 3 & \frac{b+2c}{\delta} \\ \frac{b+2c}{\delta} & \frac{2b+c}{\delta} & 3 \end{bmatrix}.$$

Linear Algebra and its Applications, 433, 893-907, (2010)

2011: 21st IWMS, Tartu, Estonia

2011: 21st IWMS, Tartu, Estonia

2011: 2011: PROBASTAT 2011 Smolenice, Slovak Republic

Julia Voulafova and Steve Kirkland

2011: PROBASTAT 2011 Smolenice, Slovak Republic

Honouring 70th Birthday

2011: MatTriad, Tomar, Portugal

Invited speaker.

Discussions with Prof Ivo Marek
leading an invitation to speak
at a conference in his honour
in Prague in 2013.

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2011: MatTriad, Tomar, Portugal

2012: International Workshop and Conference on Combinatorial Matrix Theory and Generalized Inverses of Matrices, Manipal University, India

2012: The Derivation of Markov Chain Properties using Generalized Matrix Inverses

A book chapter giving a survey of the the application of generalized inverses to stationary distributions, moments of first passage time distributions and moments of occupation time random variables in Markov chains.

"*Lectures on Matrix and Graph Methods*" (pp 61-89) Ravindra B. Bapat, Steve Kirkland, K. Manjunatha Prasad and Simo Puntanen (Editors), Manipal University Press, Manipal, Karnataka, India. ISBN: 978-81-922759-6-3 (2012).

2012: Markov chain properties in terms of column sums of the transition matrix

Questions are posed regarding the influence that the column sums of the transition probabilities of a stochastic matrix (with row sums all one) have on the stationary distribution, the mean first passage times and the Kemenyconstant of the associated irreducible discrete time MC.

Some new relationships, including some inequalities, and partial answers to the questions, are given using a special generalized matrix inverse that has not previously been considered in the literature on MCs.

Acta et Commentationes Universitatis Tartuensis de Mathematica. 16, (1), 33-51, (2012)

2012: Markov chain properties in terms of column sums of the transition matrix

We use the generalised inverse $H \equiv [h_{ij}] \equiv [I - P + \boldsymbol{e}\boldsymbol{c}^T]^{-1}$ where

$\boldsymbol{c}^T = (c_1, c_2, \dots c_m)$ is the row vector of column sums, $\boldsymbol{c}^T \boldsymbol{e} = m.$

$$\pi_j = \sum_{i=1}^{m} c_i h_{ij}, h_{i\bullet} \equiv \sum_{j=1}^{m} h_{ij} = 1/m,\ \boldsymbol{c}^T H \boldsymbol{e} = 1.$$

$$\boldsymbol{c} = \boldsymbol{e} \Leftrightarrow \boldsymbol{\pi} = \boldsymbol{e}/m,\ m_{ij} = \begin{cases} \dfrac{1}{\pi_j} = \dfrac{1}{\sum_{i=1}^{m} c_i h_{ij}}, & i = j, \\[2ex] \dfrac{h_{jj} - h_{ij}}{\pi_j} = \dfrac{h_{jj} - h_{ij}}{\sum_{i=1}^{m} c_i h_{ij}}, & i \neq j. \end{cases}$$

$$\sum_{i=1}^{m} m_{ij} - \sum_{i=1}^{m} c_i m_{ij} = m - \frac{c_j}{\pi_j} = m - c_j m_{jj}.$$

$$K = 1 - (1/m) + tr(H) = 1 - (1/m) + \sum_{j=1}^{m} h_{jj}.$$

Acta et Commentationes Universitatis Tartuensis de Mathematica. 16, (1), 33-51, (2012)

2012: Haifa Matrix Theory Conference, Technion, Haifa, Israel

Commemorated Prof. Michael (Miki) Neumann and Prof. Uriel Rothblum. Invited speaker at the Special session in memory of Professor Michael (Miki) Neumann. Spoke on “Generalized inverses of Markovian kernels in terms of properties of the MC”.

Connecticut (2006)

IWMS (2004)

2013: PIM Conference in honour of Ivo Marek, Prague, Czech Republic

Invited speaker at the conference on "Preconditioning of Iterative Methods".

2013: 22nd IWMS , Toronto, Canada

Special session in honour of Shayle Searle

2013: 22nd IWMS , Toronto, Canada

2013: The Distribution of Mixing Times in Markov Chains

Expressions for the probability generating function, and hence the probability distribution of the mixing time, starting in state i, are derived and special cases explored.

This extends the results the LAA papers of (2006) and (2008).

Let $T_i^{(0)}$ be the first hitting time of the mixing state *i*.

Let $T_i^{(0)}$ be the first passage time of the mixing state *i*.

$$f_{i,n} = P\{T_i^{(0)} = n \mid X_0 = i\},\ f_i(s) = \sum_{n=0}^{\infty} f_{i,n} s^n.$$

$$g_{i,n} = P\{T_i^{(1)} = n \mid X_0 = i\},\ g_i(s) = \sum_{n=1}^{\infty} g_{i,n} s^n \left[P_{ij}(s) \right].$$

$$P_{ij}(s) = \sum_{n=0}^{\infty} p_{ij}^{(n)} s^n,\ f_i(s) = \sum_{j=1}^{m} \pi_j \left[\frac{P_{ij}(s)}{P_{jj}(s)} \right],\ g_i(s) = f_i(s) - \frac{\pi_i}{P_{ii}(s)}.$$

Asia-Pacific Journal of Operational Research. Volume 30 (1), 29pp. (2013)

2013: The Distribution of Mixing Times in Markov Chains

In addition, some new explicit results for the distribution of the recurrence and the first passage times in general irreducible two and three state MCs are presented together with explicit distributions of the hitting time and mixing time random variables.

Asia-Pacific Journal of Operational Research. Volume 30 (1), 29pp. (2013)

2014: 23th IWMS Ljubljana, Slovenia

2014: Generalized inverses of Markovian kernels in terms of properties of the Markov chain

All one-condition generalized inverses of the Markovian kernel $I - P$, where P is the transition matrix of a finite irreducible MC, can be uniquely specified in terms of the stationary probabilities and the mean first passage times of the underlying MC.

Special sub-families include the group inverse of $I - P$, Kemeny and Snell's fundamental matrix of the MC and the Moore-Penrose g-inverse. The elements of some sub-families of the generalized inverses can also be re-expressed involving the second moments of the recurrence time variables.

Some applications to Kemeny's constant and perturbations of MCs are also considered.

Linear Algebra and its Applications, 447, 38-55 (2014)

2014: Generalized inverses of Markovian kernels in terms of properties of the Markov chain

Let $G = G(\boldsymbol{\alpha}.\boldsymbol{\beta},\gamma)$ be any g-inverse of $I - P$. Then the elements of $G = [g_{ij}]$ can be expressed in terms of the parameters $\{\alpha_i\},\{\beta_j\},\gamma$, the stationary probabilities $\{\pi_j\}$, and the mean first passage times $\{m_{ij}\}$, of the MC, with $\delta_j \equiv \sum_{k \neq j}^{m} \beta_k m_{kj}$, as

$$g_{ij} = \begin{cases} \left(1+\gamma+\delta_j - m_{ij} + \sum_{k\neq i} \pi_k \alpha_k m_{ik} - \sum_{k=1}^{m} \pi_k \alpha_k \delta_k\right)\pi_j, & i \neq j, \\ \left(1+\gamma+\delta_j + \sum_{k\neq j} \pi_k \alpha_k m_{jk} - \sum_{k=1}^{m} \pi_k \alpha_k \delta_k\right)\pi_j, & i = j. \end{cases}$$

Let $A^{\#} = \left[a_{ij}^{\#}\right] = [I - P + \boldsymbol{e}\boldsymbol{\pi}^T]^{-1} - \boldsymbol{e}\boldsymbol{\pi}^T$ be the group inverse of $I - P$, then $a_{ij}^{\#} = \pi_j(\tau_j - 1 - m_{ij})$ for $i \neq j$; with $a_{jj}^{\#} = \pi_j(\tau_j - 1)$. ($\tau_j \equiv \sum_{k=1}^{m} \pi_k m_{kj}$).

Linear Algebra and its Applications, 447, 38-55 (2014)

2014: The role of Kemeny's constant in properties of Markov chains

In a finite irreducible MC with stationary probabilities $\{\pi_i\}$ and mean first passage times m_{ij} (mean recurrence time when $i = j$) it was first shown, by Kemeny and Snell (1960) that $\sum_j \pi_j m_{ij}$ is a constant, K, (Kemeny's constant) not depending on i.

A variety of techniques for finding expressions and bounds for K are given.

The main interpretation focuses on its role as the expected time to mixing in a MC. Various applications are considered including perturbation results, mixing on directed graphs and its relation to the Kirchhoff index of regular graphs.

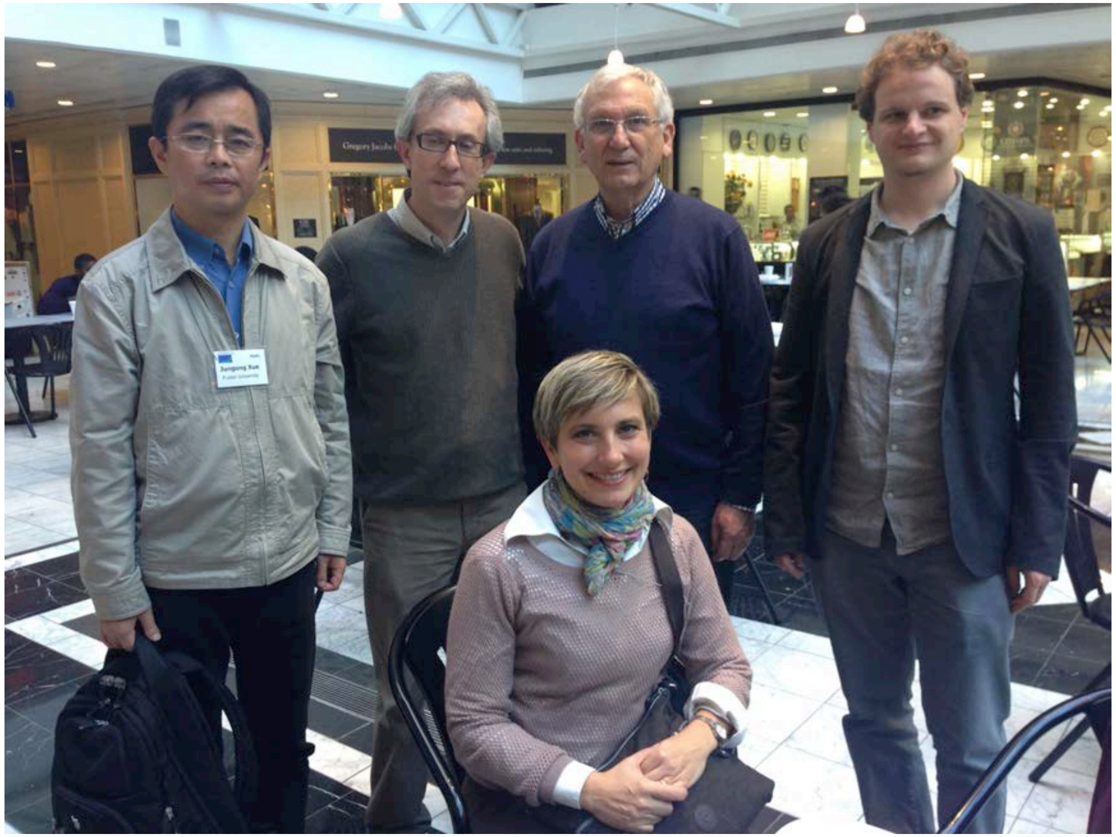
Communications in Statistics – Theory and Methods. 43: 1 - 13, (2014)

2015: Chair IOC 24th IWMS, Hainan

2015: SIAM - Applied Linear Algebra Conference, Atlanta, Georgia, USA

Invited speaker in a Featured Mini **s**ymposium on “Numerical Methods for Markov Chains and Stochastic Models”

Speakers:



2016: Accurate computation of mean first passage times in Markov and Markov renewal processes

EGTH Algorithm

Step 1(i): Start with $P^{(N)} = \left[p_{ij}^{(N)} \right]$, carry out the GTH algorithm by calculating successively,

for $n = N, N-1, \ldots, 2$, $p_{ij}^{(n-1)} = p_{ij}^{(n)} + \dfrac{p_{in}^{(n)} p_{nj}^{(n)}}{S(n)}$, $1 \le i \le n-1, 1 \le j \le n-1$, where $S(n) = \sum_{j=1}^{n-1} p_{nj}^{(n)}$.

(Note that we only have to retain the $p_{in}^{(n)} (1 \le i \le n-1)$ and $p_{nj}^{(n)} (1 \le j \le n-1)$, i.e. the n-th row and n-th column of $P^{(n)}$ for $n = 2, \ldots, N$, as in the GTH algorithm.)

Step 1(ii): Start with the mean holding time vector $\boldsymbol{\mu}^{(N)T} = (\mu_1^{(N)}, \mu_2^{(N)}, \ldots, \mu_{N-1}^{(N)}, \mu_N^{(N)})$ and calculate successively for $n = N,\ N-1, \ldots, 2$, $\mu_i^{(n-1)} = \mu_i^{(n)} + \dfrac{\mu_n^{(n)} p_{in}^{(n)}}{S(n)}$, $1 \le i \le n-1$.

Step 1(iii): Calculate the $N \times 1$ column vector $\boldsymbol{m}_N^{(1)(N)} = (m_{i1})$, where $m_{11} = \mu_1^{(1)}$, $m_{21} = \dfrac{\mu_2^{(2)}}{S(2)}$, and for $i = 3, \ldots, N$, $m_{i1} = \dfrac{\mu_i^{(i)} + \sum_{k=2}^{i-1} p_{ik}^{(i)} m_{k1}}{S(i)}$.

This gives the entries of the first column of $M = [m_{ij}]$, i.e. $\boldsymbol{m}_N^{(1)(N)}$ where $M = \left(\boldsymbol{m}_N^{(1)(N)}, \boldsymbol{m}_N^{(2)(N)}, \ldots, \boldsymbol{m}_N^{(N)(N)}\right)$ with $\boldsymbol{m}_N^{(1)(N)T} = (m_{11}, m_{21}, \ldots, m_{N1})$.

2016: Accurate computation of mean first passage times in Markov and Markov renewal processes

Step 2: For $k = 2, 3, 4, \ldots, N-1, N$.

(i) Repeat Step 1(i) but with $P^{(N)} = P^{(N)(k)}$ where $P^{(N)(k)} = R_1^{(N)} P^{(N)(k-1)} C_1^{(N)}$ with $P^{(N)(1)} = P^{(N)}$

(Comment: This steps leads to the appropriate $p_{in}^{(n)}$ and $p_{nj}^{(n)}$ elements.)

(ii) Repeat Step 1(ii) but with $\boldsymbol{\mu}^{(N)} = \boldsymbol{\mu}^{(N)(k)}$ where $\boldsymbol{\mu}^{(N)(k)T} = \boldsymbol{\mu}^{(N)(k-1)T} C_1^{(N)}$ with $\boldsymbol{\mu}^{(N)(1)} = \boldsymbol{\mu}^{(N)}$

(Comment: This step leads to the appropriate $\mu_i^{(n)}$ elements. In the case of a MC no permutation of the elements is required, since $\mu_i^{(N)} = 1$ for all i.)

(iii) Repeat Step 1(iii) to calculate the $N \times 1$ column vector $\bar{\boldsymbol{m}}_N^{(k)(N)}$ where

$\bar{\boldsymbol{m}}_N^{(k)(N)T} = (m_{kk}, m_{k+1,k}, \ldots, m_{Nk}, m_{1k}, \ldots, m_{k-1,k})$.

Step 3: Combine the results of the Steps 1(iii) and 2(iii) to find M as follows.

Let $\overline{M} = (\boldsymbol{m}_N^{(1)(N)}, \overline{\boldsymbol{m}}_N^{(2)(N)} \ldots, \overline{\boldsymbol{m}}_N^{(N)(N)})$ and reorder the elements of $\overline{M}$ to obtain M = $(\boldsymbol{m}_N^{(1)(N)}, \boldsymbol{m}_N^{(2)(N)} \ldots, \boldsymbol{m}_N^{(N)(N)})$.

Special Matrices **4**:151-175 (2016).

2016: Why the Kemeny Time is a Constant – with Karl Gustafson

(with Gustafson K.) Special Matrices **4**:176-180 (2016)

2016: Why the Kemeny Time is a Constant – with Karl Gustafson

We present a new fundamental intuition for why the Kemeny feature of a MC is a constant. This new perspective has interesting further implications.

The new intuition is to see the well-known basic mean first passage time matrix equation $M\boldsymbol{\pi} = K\boldsymbol{e}$ as a change-of-basis procedure. By viewing M with its diagonal elements removed as the change-of-basis matrix to the natural basis intuits that one must "end up with equally probable pure states".

$\overline{M}\boldsymbol{\pi}$ is in the principal eigenspace of P and is therefore is a constant times $\boldsymbol{e}$. We then explain why the Kemeny vector has equal coordinates.

(with Gustafson K.) Special Matrices **4**:176-180 (2016)

2016: The Computation of the Stationary Distribution, the Mean First Passage Times and the Group Inverse and associated with a Markov Chain via Perturbations

By using perturbation techniques, starting from a simple transistion matrix where only simple derivations are formally required for a generalized inverse, the stationary distribution and the mean first passage time matrix, we update a sequence of matrices, formed by linking the solution procedures via generalized matrix inverses and utilising matrix and vector multiplications.
Six different algorithms are given, some modifications are discussed, and numerical comparisons are made using a test example.

Submitted to Linear Algebra and its Applications, AriXiv.com (2016)

Thanks for the memories

I would like to thank the organisers

– Simo and the IOC team,

– Francisco and the LOC team for this Workshop.

I would like to thank all those who in many ways have enriched my experiences as an academic.

I would like to thank all the members of the IOC.

I wish them well in future and may the IWMS series of

Workshops – wonderfully conceived by George Styan –

continue to make an impact in the area of “Matrices and Statistics”.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